Intro Video: Section 4.7
Constrained Optimization

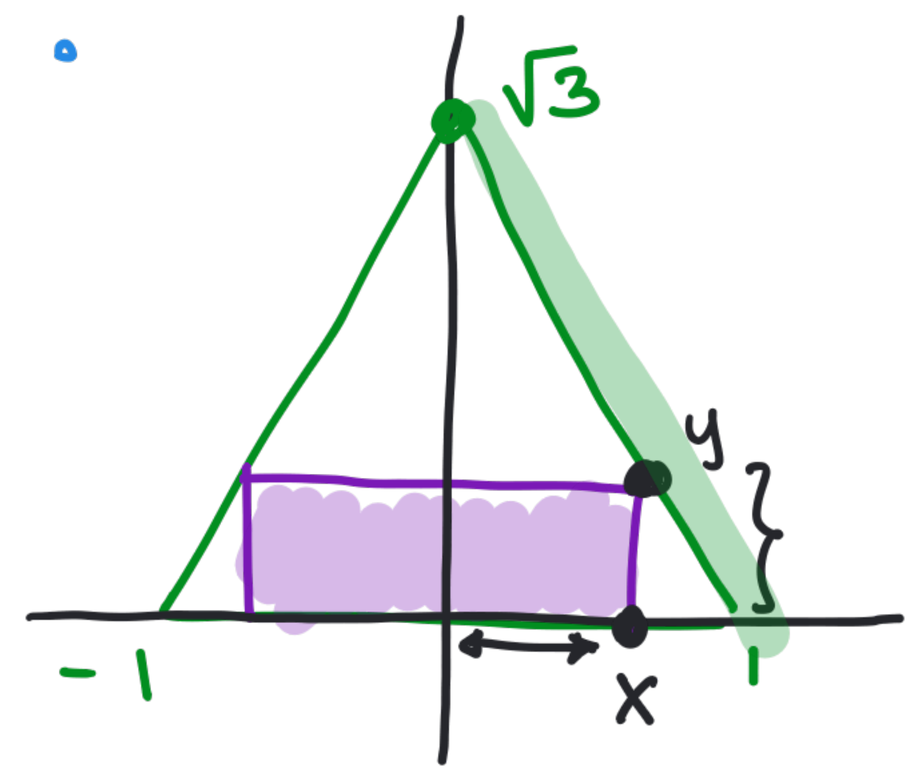
Math F251X: Calculus I

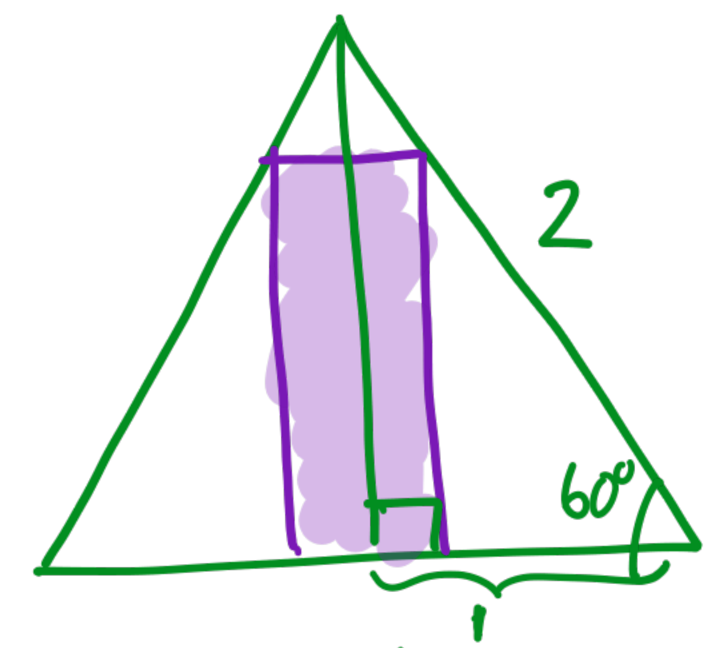
What is constrained/applied optimization?

Example: What are the dimensions of the largest rectangle inscribed in an equilateral triangle of side length 2?

→ Maximize area

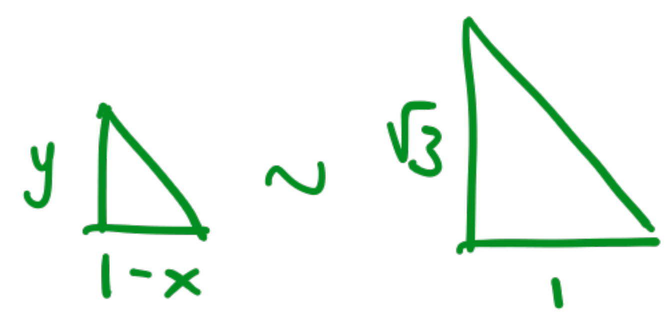
→ Constraint: the rectangle has to fit into the triangle.





$A = 2xy$

$\boxed{\text{Constraint}}$

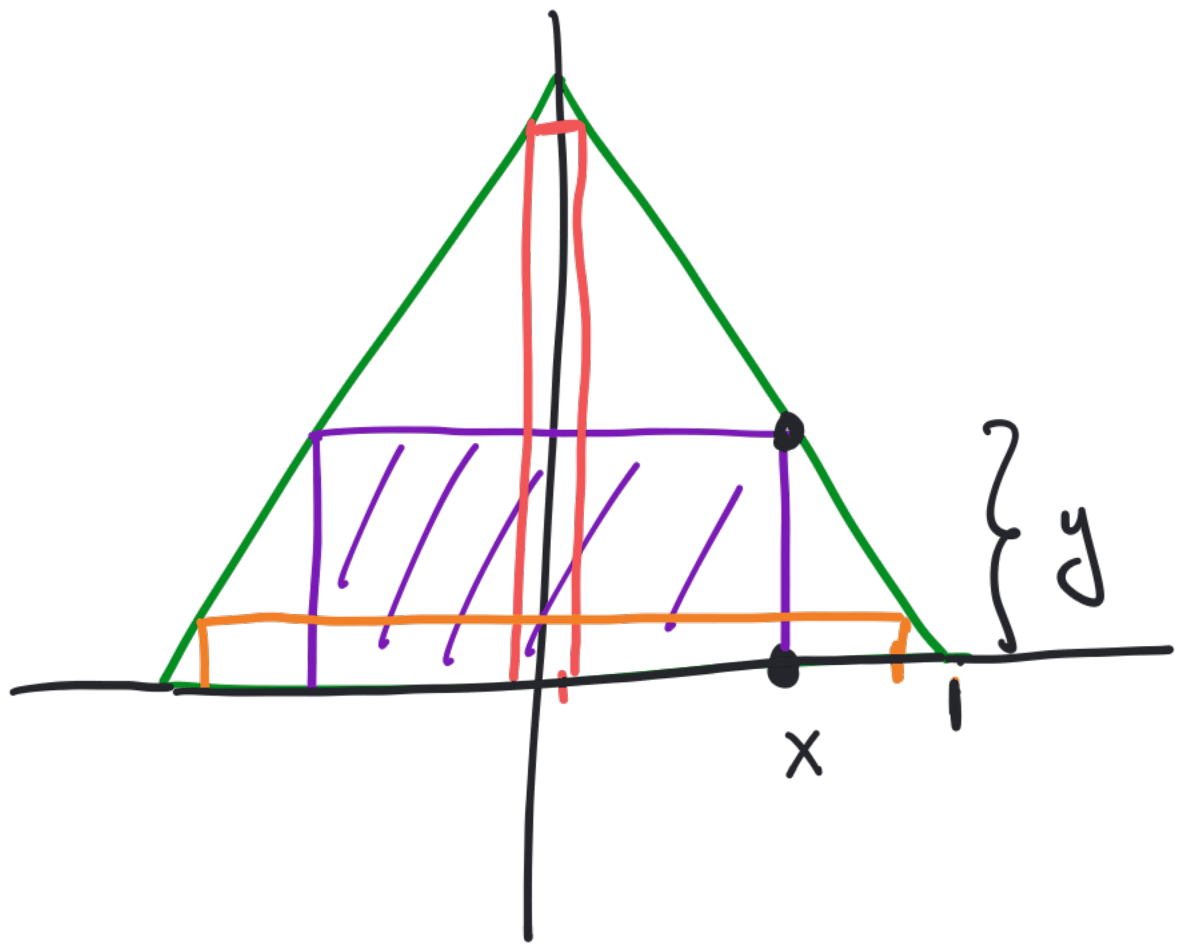


$y = \sqrt{3}(1-x)$

$\frac{\text{height}}{2} = \sin(60°) \implies$

$\text{height} = \sqrt{3}$

$$\boxed{A(x) = 2x\left(\sqrt{3}(1-x)\right)}$$



$A(x) = 2x\left(\sqrt{3}(1-x)\right)$
$= (2\sqrt{3})(x - x^2)$

Domain: $[0,1]$

$\rightarrow$ This allows $x=0$ and $y=1$ to be valid dimensions of a "degenerate" rectangle with area 0

Where does $A(x)$ attain an absolute maximum on $[0,1]$?

$\boxed{\text{Calculus!}}$

$A'(x) = 2\sqrt{3}(1-2x)$

$A'(x)$ never DNE

$A'(x) = 0 \Rightarrow 1-2x = 0$
$\Rightarrow \boxed{x = 1/2}$

Is $x = 1/2$ an absolute maximum?

Is it a local max? $A''(x) = -4\sqrt{3} < 0$

$x = 1/2$ is a local max! at $x=0$, $x=1$, $A(x) = 0$.

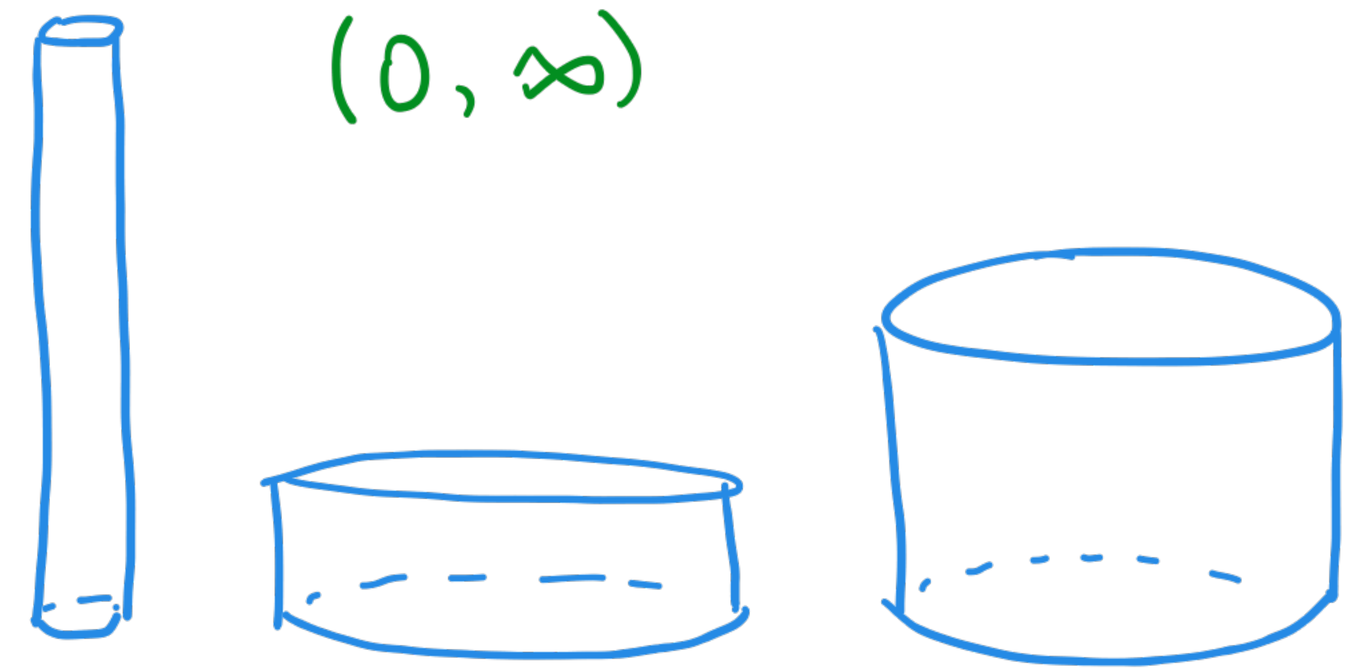
So $x = 1/2$ is an absolute max. $\boxed{\text{ANSWER}}$ Dimensions are $1 \times \frac{\sqrt{3}}{2}$.

When $x = 1/2$, $y = \sqrt{3}(1 - 1/2) = \frac{\sqrt{3}}{2}$

Example: A storage tank is designed in the shape of a right circular cylinder, with fixed volume of 1000L. What dimensions minimize the amount of metal needed?

DOMAIN?

$(0, \infty)$



Minimize: Surface area

Constraint: volume is fixed.

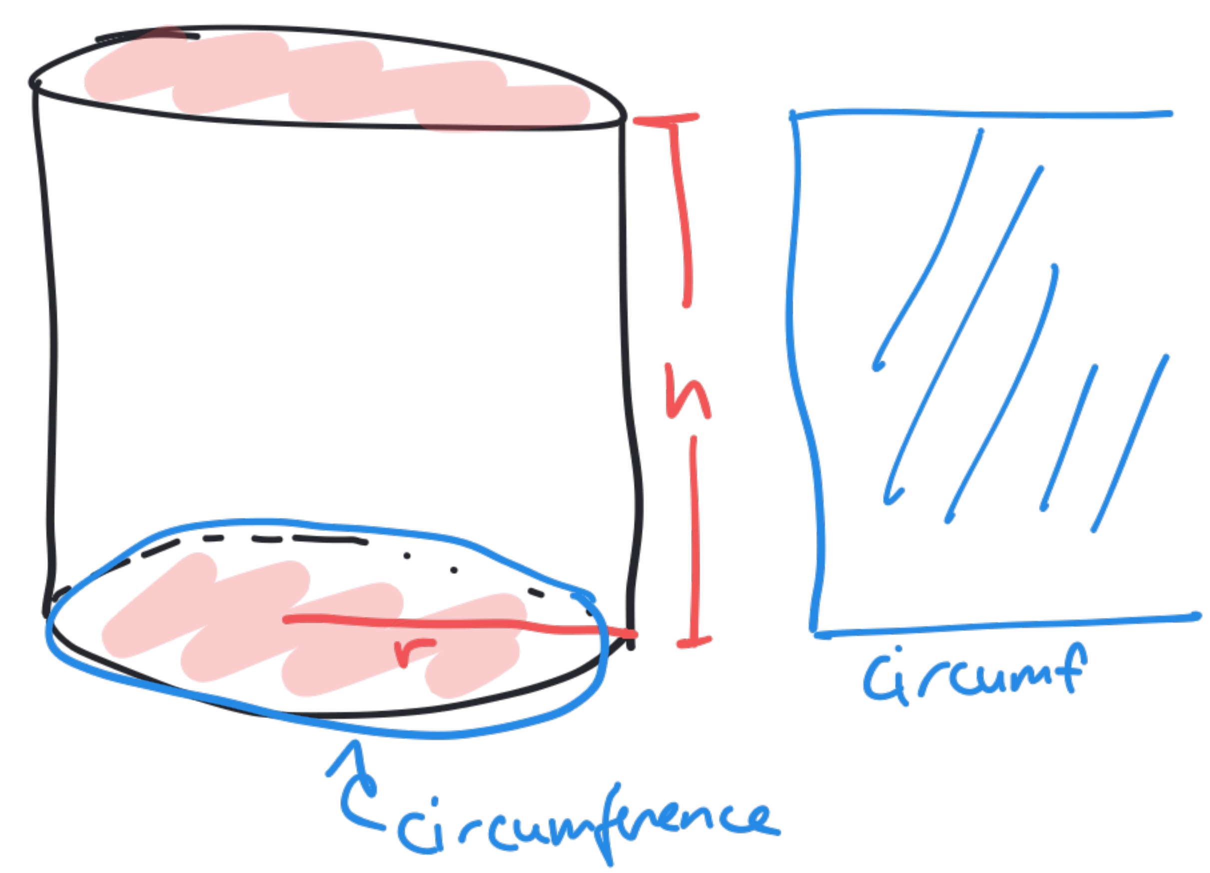
Set up variables.

$$V = (\pi r^2)(h) = 1000 \Rightarrow h = \boxed{\frac{1000}{\pi r^2}}$$

$$SA = 2(\pi r^2) + 2\pi r h$$

$$SA(r) = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$



$$SA(r) = 2\pi r^2 + \frac{2000}{r}$$ ← minimize r

$$SA'(r) = 4\pi r - \frac{2000}{r^2}$$

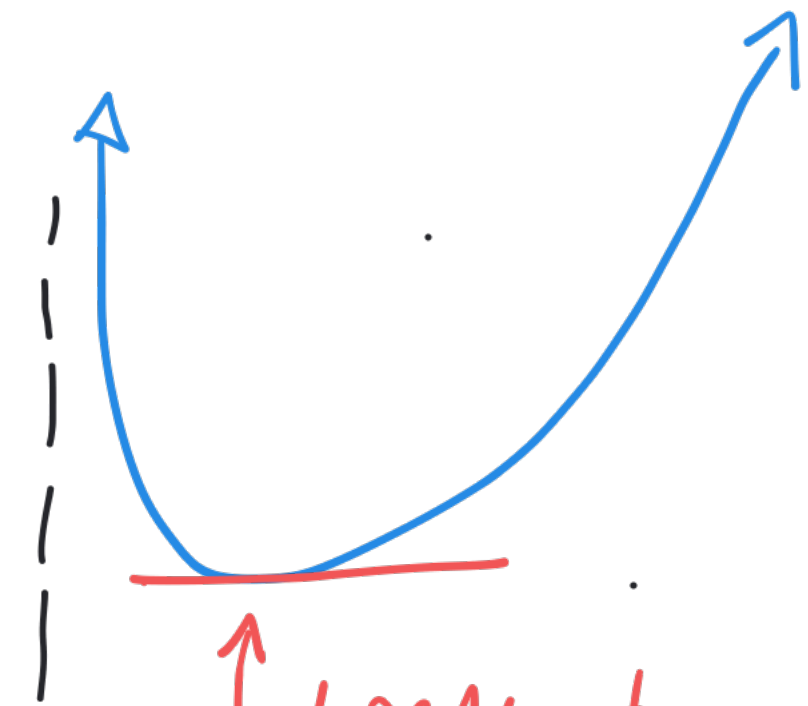
Note $SA'(r)$ DNE at $x=0$, which is not in our domain.

$$SA'(r) = 0 \Rightarrow 4\pi r = \frac{2000}{r^2} \Rightarrow r^3 = \frac{2000}{4\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{2000}{4\pi}} = 10\sqrt[3]{\frac{1}{2\pi}} = \frac{10}{\sqrt[3]{2\pi}} \approx 5.41$$

$$SA''(r) = 4\pi + \frac{4000}{r^3} > 0 \quad \text{when } r > 0$$

So SA is always CU for $r > 0$!



↑ LOCAL & ABSOLUTE MIN on domain!

$\boxed{\text{Answer}}$ the dimensions that minimize surface area are $r = \frac{10}{\sqrt[3]{2\pi}}$

and $$h = \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{2\pi}}\right)^2} = \frac{10\cancel{00}}{\frac{1\cancel{00}\,\pi}{\sqrt[3]{2\pi}}} = \frac{10\sqrt[3]{4\pi^2}}{\pi} = \frac{10\sqrt[3]{4}\,\pi^{2/3}}{\pi^{3/3}} = \frac{10\sqrt[3]{4}}{\sqrt[3]{\pi}} = 10\sqrt[3]{\frac{4}{\pi}}$$

About $r = 5.42$, $h = 10.84$

A Framework for Approaching Optimization

1. Read the problem two or three times. Draw pictures. Label them. Pick specific numerical examples, to make the problem concrete. Be creative. Try more than just one approach.
2. Identify the quantity to be minimized or maximized (and which one... min or max).
3. Chose notation and explain what it means.
4. Write the thing you want to maximize or minimize **as a function of one variable**, including a reasonable **domain**.
5. Use calculus to answer the question and justify that your answer is correct.